Percolation: A Model of Critical Phenomena And Its Application to High-Energy Nuclear Collisions

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1 Introduction

The percolation model is a simple and versatile tool that describes the interaction of multiple phases and critical phenomena. With a wide variety of applications in many fields it can be used to characterize physical processes in phase transitions such as sea-ice, metal-insulator transitions, and high-energy nuclear collisions. The quark-gluon plasma is a state of matter predicted by quantum chromodynamics. This exciting prediction has not yet been conclusively observed in experiment. There are still uncertainties as to exactly what a transition within colliding nuclei to this new state would lool like experimentally. Current theoretical predictions suggest that fluctuations will not mask a sharp change in the value of some experimental observables. However, many assumptions are made in the models that result in these collisions. Many of these assumptions may neglect the fluctuations that would make the phase transition less drastic.

With the goal of understanding the application of percolation to nuclear matter, a brief introduction of the terminology and types of percolation models will be made in section 2. In section 3 , we will study a very simple two dimensional model and observe how the relavent percolation parameters characterize the behavior of the system. There are two variations of this toy model. In the first, discs will be uniformly distributed on a larger disc. In the second, the discs will be distributed according to the density of a sphere projected onto a plane. Understanding the application to colliding nuclei requires a brief introduction to nuclear physics and the quark-gluon plasm to be made in section 4.1. Applying these physical principles to a variation of the model discussed in section 3, we will then examine the implications of percolation on experimental observables and suggest ways to refine the model to make better predictions.

2 Percolation Theory

The general percolation formulation concerns a number of elementary objects placed according to some random distribution in a d-dimensional space. In this space there is a well-defined maximum radius within which these objects can communicate. Percolative processes may be divided into two basic categories, lattice or discrete percolation and continuum percolation.

2.1 Lattice Percolation

Lattice percolation characterizes systems with a well-constrained geometrical structure. These include the interaction of atoms in crystals. Percolation may model the behavior of the bonds between sites or the state of the site itself. If each site which has a possibility of being in one of two states is being modelled, and each site has probability p of being occupied, then a cluster is defined as a group of nearest neighbors with the same state. At some probability, p_c, there is a guaranteed one cluster that will span the volume. The is called the *critical probability*, and depends on the geometry of the system and whether it is a system of sites or bonds. Let $n_s(p)$ be the number of clusters with exactly s sites per unit volume. There are a set of universal critical exponents $\{\alpha, \beta, \gamma, \delta, \nu\}$ that describe the behavior of the system near the critical probability [1]. Here the $\hat{S}_x$ operator is one if its operand is non-analytic in x and zero otherwise.

$$\hat{S}_p \sum_{\infty}^{s=1} n_s(p) \propto |p - p_c|^{2-\alpha} \tag{1}$$

$$\hat{S}_p \sum_{\infty}^{s=1} s n_s(p) \propto |p - p_c|^{\beta} \tag{2}$$

$$\hat{S}_p \sum_{\infty}^{s=1} s^2 n_s(p) \propto |p - p_c|^{-\gamma} \tag{3}$$

$$\hat{S}_H \sum_{\infty}^{s=1} s n_s(p_c)\, e^{-Hs} \propto H^{1/\delta} \tag{4}$$

$$\eta(p) \propto |p - p_c|^{-\nu} \tag{5}$$

Near the critical probability the macroscopic "conduction" of the system obeys the characteristic behavior

$$\sigma_{dc} \propto (p - p_c)^{\mu}\, \theta(p - p_c) \tag{6}$$

where μ depends only on the geometry.

Another userful mathematical property of lattice percolation systems capitalizes on the complimentary properties of lattice geometries to define *matching* lattices shown in Figure 2.1. Two lattices are *matching* if there exists a one-to-one correspondence between the bonds of the two lattices and if "conduction" across one bond implies the corresponding bond is blocked in the other lattice. This enables p_c to be calculated exactly for many geometries.

As a simple example, the familiar Ising model describes the interaction of magnetic dipoles fixed on a two-dimensional lattice. Here the two phases are spin up and spin

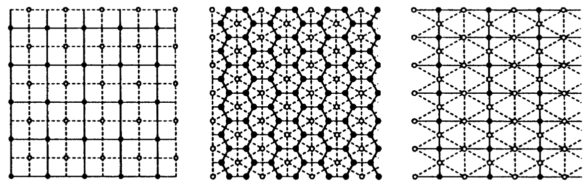

Figure 1: Shown are three pairs of matching lattices. One lattice is drawn with solid lines and filled circles and its matching lattice with dashed lines and empty circles.

down. The iteraction of two sites is such that they are at a lower energy when their spins are aligned. If the system begins with a random distribution of spin up and spin down states, the temperature of the system will determine the long-term behavior of the system. At high temperatures, the distribution of states will remain random; however, if the temperature lowers such that the interaction of sites dominates, these two phases will compete for the entire system. As the system evolves one state will dominate the system even in the absense of an external magnetic field.

2.2 Continuum Percolation

Continuum percolation describes the interaction of objects not restricted to points on a lattice. This mathematical problem was originally developed to illustrate the flow of liquid through a porous medium [1]. In Section 3 will make a thorough examination of a two-deminsional system. We will now brifly address a current application of continuum percolation. Three dimensional continuum percolation models are now being used to explain the enormous magnitude of colossal magnetoresistance in low-T_C manganites [3]. Most of the evidence suggests that manganite characteristics are largely governed by competition of double-exchange ferromagnetism and Jahn-Teller polaron formation. However, these cannot explain the large magnetoresistance. Percolation studies indicate competition between two ferromagnetic and charge-ordered ground states better characterizes the behavior. The dominance of these two phases is determined by both the temperature of the sample and the relative concentrations of its components. The average ionic radius is determined by this relative composition and when plotted against the Curie temperature displays the characteristic percolation "S" curve to be discussed in section 3.

3 Toy Model

We will examine a two-dimensional continuum system, the random distribution of equal size discs within a larger disc. One might imagine tossing coins randomly in a larger ring. If the placement of the tosses is relatively uniform, at first each toss will result in several coins scattered across the area of the disc with only a few coins overlapping. With more and more tosses many small groups of overlapping coins called clusters will develop.

Figure 2: In this Toy Model small discs are cast into a larger disc uniformly. Show are three snap shots of the formation of clusters as more and more discs are distributed.

Further tosses will begin to connect these small clusters together to form larger and larger clusters. Eventually, one large cluster that spans the diameter of the disc will form with relatively few tosses. It is this rapid development of such a large cluster that underlies many critical phenomena. Figure 3 demonstrates this process in a computer model whose small discs are one-hundredth the area of the larger. As larger clusters form to finally combine in one large cluster the area of the largest cluster develops characteristically for the geometry of the system. For this model, the development of this area is shown in Figure 3 as the characteristic “S” curve.

The relative size of the smaller discs is a parameter of great importance. This parameter determines how sharp a transition is and determines the fluctuations of the transition from one occurrence to the next. The development of the largest cluster is demonstrated in Figure 3 of a system with the size of the smaller discs 1/400 that of the larger. The area of the largest cluster is plotted against the overall density of discs with the smaller discs normalized to unity and the derivative is plotted at one-tenth its scale. Note that the derivative peaks at a density of 1.05. This is the percolation point and its associated critical density.

As the size of the smaller discs becomes smaller this transition becomes sharper, and in the limit of the ratio of the radii approaching zero, the transition becomes a step function. Figure 3 demonstrates the increased slope of the area of the largest cluster when the ratio of radii is five-thousandths. It was well know from Monte-Carlo studies that extrapolation in the limit of an infinitesimal radius gave a critical density of 1.175, but it was shown analytically in 1990 [2]. As shown in Figure 5, this independent analysis suggests a limit of 1.14. This is an important check that the basic procedure and definitions are consistent, and while the difference is not negligible these data are limited by statistics and many of the first estimates in the literature suggested a slightly lower value.

Another important observation to make from this simple model is the fluctuations in the transition. One should note that while the percolation point is well defined for these processes, the density at which the area of the largest cluster makes it most dramatic increase is not the same for every sequence of tosses. This dependence in the magnitude of these fluctuations on the ratio of radii are shown in Figure 6. With increasing relative

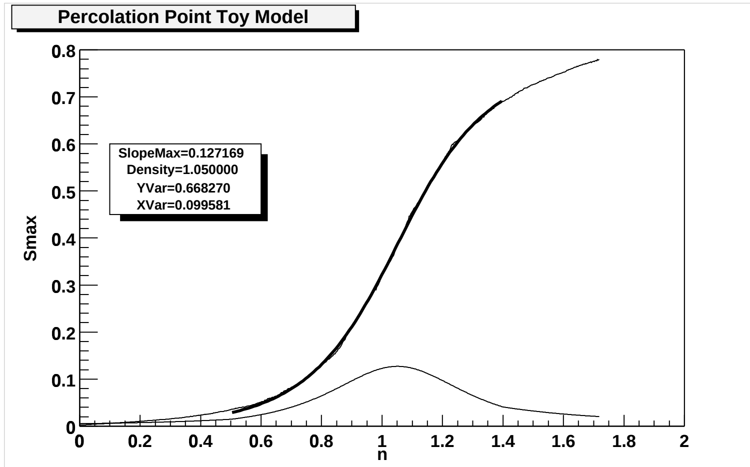

Figure 3: The ratio of the volume of the largest cluster to the total volume of the system, Smax, is plotted against n, the total number of smaller discs per total volume of the system. The "S" curve is characteristic to the two dimensional spheres geometry and for the ratio of the volume of the smaller spheres to the volume of the system is 1/400. Also plotted is the slope of the "S" curve which shows a dramatic peak at the percolation point.

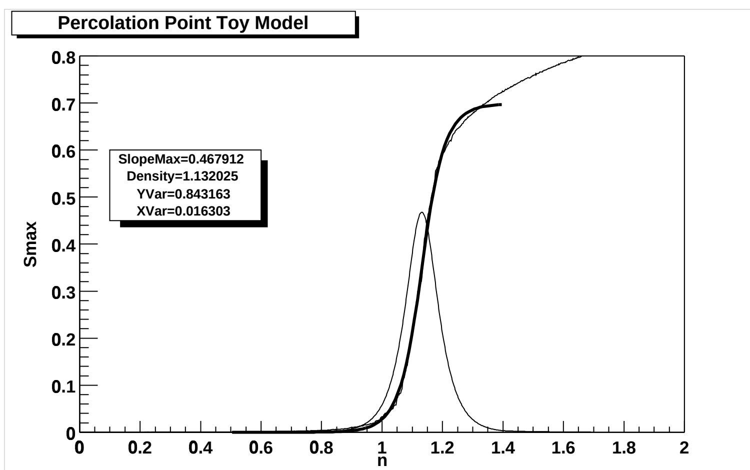

Figure 4: The ratio of the volume of the largest cluster to the total volume of the system, Smax, is plotted against the n, the total number of smaller discs per total volume of the system. The "S" curve is characteristic to the two dimensional spheres geometry. Comparied to Figure 3 the development of the area of the largest cluster is much sharper. In the limit that the ratio of the smaler volume to the larger volume approaches zero, the "s" curve becomes a step funtion. And the slope of the "S" curve also plotted becomes a delta function. This is the percolation point and the associated critical density.

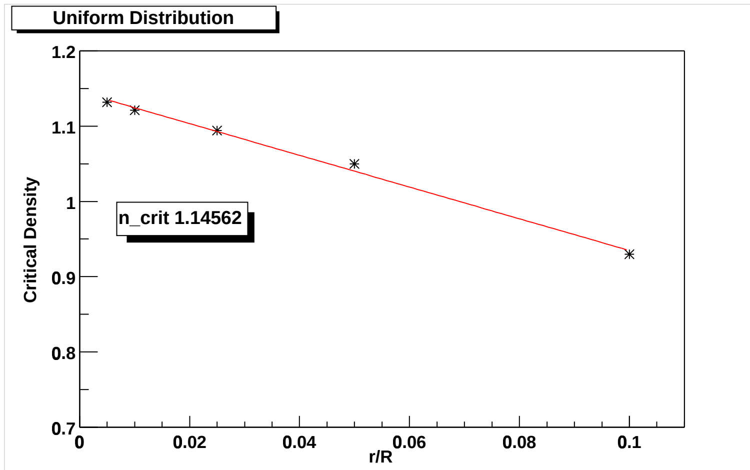

Figure 5: The critical density, n_crit is the total number of participants per system volume at which the slope of the area of the largest cluster is a maximum. The ratio of the radii of the unit domain and the total system area is r/R. In the limit of $r/R \to 0$, the critical density approaches 1.15.

size not only is the transition softened but for a given transition the percolation point could occur at drastically different densities.

4 Nuclear Phase Transitions

The percolation model may be applied to high-energy nuclear physics, specifically, the collision of two nuclei. One of the major recent experimental goals of this field has been to verify a theoretical prediction of a new state of matter called the quark- gluon plasma. This new state of matter involves the breakup of protons and neutrons into their components, quarks and gluons, forming a soup of particles no longer bound in the groups of three that make up ordinary matter. This is the state of matter that would have existed shortly after the big bang and before matter condensed into hadrons, like protons and neutrons.

4.1 Theory

Now that energy densities are much lower in the universe, recreating conditions to reproduce this state in a lab require very large machines to accelerate and collide heavy ions. For a very brief time, as each nucleon collides into nucleons from the opposing ion, color strings are formed between these *wounded* nucleons. These strings are the fields of the strong force that both hold protons and neutrons together in the nucleus and bind the individual quarks that comprise the nucleons. The mediator of the strong force is the gluon which may be one of six colors: red, blue, green, and their anti-colors. As these

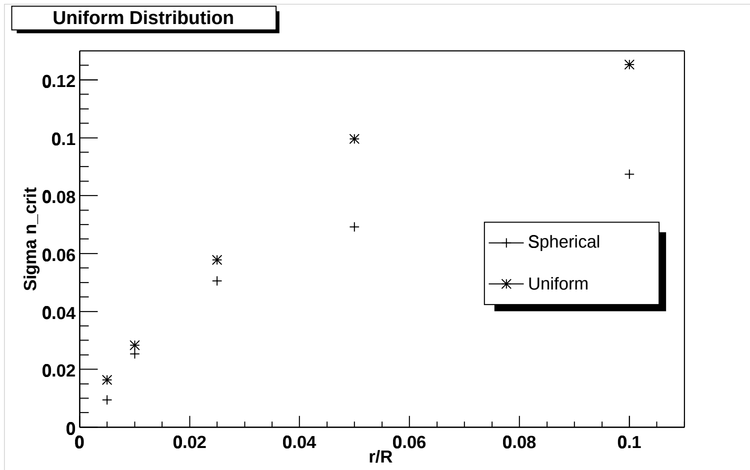

Figure 6: The fluctaions in the critical density characterized by the standard deviation of the critical density about the mean, sigma n_crit is plotted as a function of the ratio of the radii of the unit domain and the total system area, r/R, for the Toy Model with two different distributions of discs. The points showing the largest fluctuations result from uniformly distributing the smaller discs. The distribution of discs according to projection of a three dimensional sphere onto a plane correspond to the lower fluctuations.

nuclei fly past one another these strings are stretched between the nuclei between two colored objects like gluons. It is predicted that these strings will affect the production of some rare particles as the J/Ψ and the χ which are $c\bar{c}$ pairs [4]. Here c is the charm quark and $\bar{c}$ is its anti-particle. In electromagnetism positive charge placed one side of a capacitor and negative charge on the other will create an electric field between the plates that if strong enough may separate electrons from their positive nuclei. Similarly, in the collision of these nucleons a red and anti-red may be seperated in two colliding nucleons creating a field between them that if strong enough may pull apart a $c\bar{c}$ pair. If the production of these particles is reduced, experiments should be able to measure such suppression. We will next examine how percolation arguments can be made to suggest such a measurable signal, and characterize the nature of such a signal.

4.2 Application of Percolation

The parameters used to characterize percolative processes may be mapped to parameters that characterize nuclear collisions in high-energy physics experiments. For the purposes of this model, we will make several simplifications. The entire interaction will take place in the two-dimensional cross-section perpendicular to the relative velocity of the nuclei. Since the two nuclei are traveling at 0.9995 the speed of light and therefore greatly Lorentz contracted, this assumption is less crude than otherwise. The strings formed between interacting nucleons will be our unit domains. It is their density and overlap that will demonstrate the critical phenomena. In nuclear collisions the number of strings produced

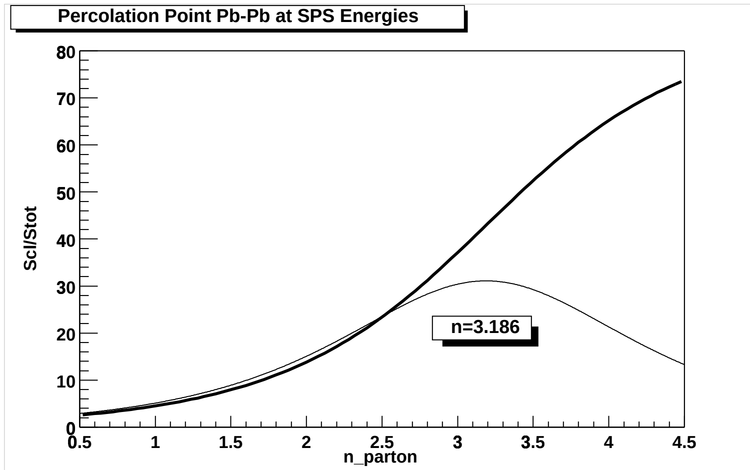

Figure 7: The ratio of the volume of the largest cluster and the total volume of the system is plotted against the average parton density for the simulation of 1000 events of two Pb nuclei colliding at SPS energies. The critical density, n_crit is the average parton density at which the slope of the area of the largest cluster is a maximum. Here the critical density is 1.15 fm^{-2}.

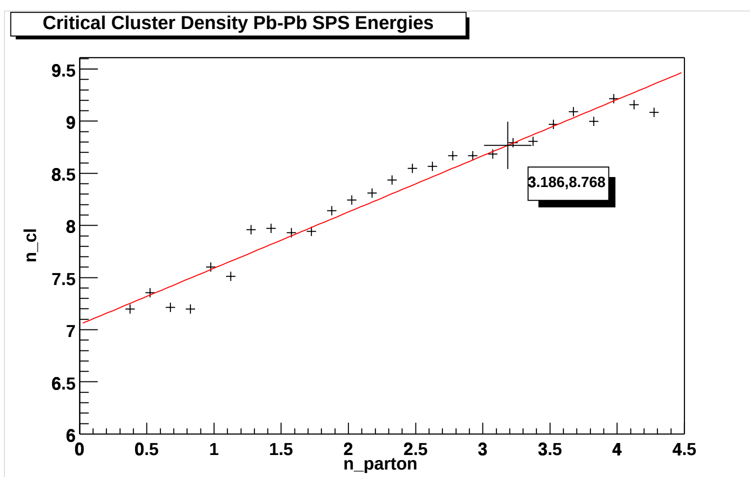

Figure 8: The critical cluster density is the density of the largest cluster at the percolation point. Plotted is the density of the largeset cluster against the average parton density. The critical cluster density is 8.77 fm^{-2}.

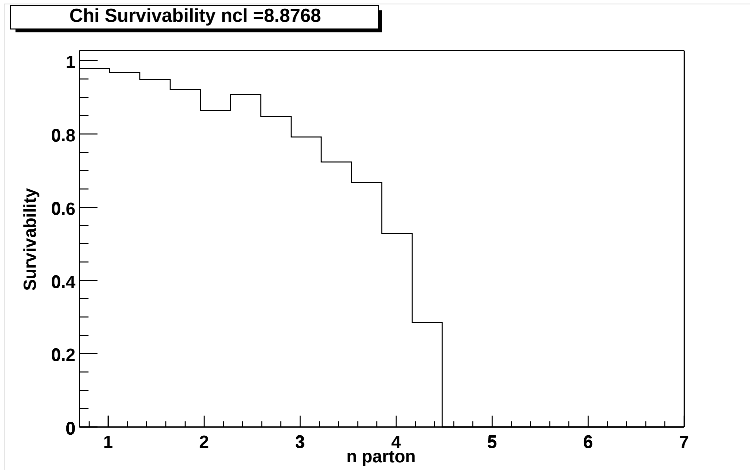

Figure 9: The ratio of the χ's that are not broken up by the quark-gluon plasma to the total is the the Survivability of the χ. The average parton density, n_parton, is directly related to the impact parameter of the colliding nuclei. The more central the collision the greater the average parton density. Thus as the average parton density increases the less likely a χ will survive.

depends on two factors. As the energy of collisions increase, the number of strings each colliding nucleon produces also increases. Second is the centrality of the collision. If the colliding nuclei graze past one-another, fewer nucleons will collide producing fewer strings. Therefore, at SPS energies there will be exactly two strings of radius 0.27 fm distributed uniformly within the volume of each wounded nucleon. The average density of a collision is therefore determined by the number of strings divided by the cross-sectional area of the interaction.

The percolation point for this model will be the average density of strings for which there is a maximum increase in the relative area of the largest cluster shown in Figure 7. The percolation point has a corresponding average density of the largest cluster shown in Figure 8. A cluster of overlapping strings will make the phase transition when the density of the cluster itself becomes greater than this critical density. As cs are formed within a cluster of density greater that the critical density, the quarks that comprise it will dissociate [4]. This process is characterized in Figure 9 which shows the probability that $c\bar{c}$ created in a collision will escape the collision area. Certainly, suppression is observed in the simulation; however, the fluctuations in the cluster density seem to smear any sharp transition. These results are by no means conclusive and further calculations still need to be performed as the model is refined. There are several modifications that should be made to enhance the model. The distribution of partons within the collision area does not depend solely on the number of wounded nucleons and may itself also be a function of the number of binary collisions. The distribution of strings within the area of the wounded nucleon may be better suited by a Poisson distribution or even determined by the overlap of the wounded nucleons. The radius of the strings is also not fixed and should follow a distribution based on experimental measurements.

5 Conclusions

In conclusion, the percolation model is adaptable to many physical processes including the predicted phase transition in hot dense hadronic matter. It is able to characterize both the critical nature of the transition and demonstrate the effects of fluctuations on parameters that can be experimentally measured. This analysis gives no conclusive indication that the suppression of the J/Ψ and χ would be obvious from a sharp onset of a quark-gluon plasma. However, there are several assumptions in this model that may be improved.

References

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